Pergamon

0021-8928(95)00052-6

J. Appl. Maths Mechs, Vol. 59, No. 3, pp. 449–457, 1995

0021-8928/95 $24.00+0.00

A MODEL OF THE DEFORMATION OF A NON-UNIFORMLY HEATED THREE-LAYER ROD WITH DELAMINATIONS†

E. I. GRIGOLYUK, Ye. A. KOGAN and V. I. MAMAI

Moscow

(*Received* 31 *May* 1994)

A complete system of relations, which govern the bending stress–strain state of non-uniformly heated three-layer rods with an asymmetric structure, a stiff compressible filler and delamination-type defects on the surfaces of contact between the carrying layers and the filler, is constructed by a variational method. These relations include the equilibrium differential equations in an invariant form for the whole domain of integration, the end conditions, the matching conditions on the boundaries of the defect-free and defective domains and elasticity relations. The initial equations of equilibrium in terms of the forces and moments are reduced to normal systems of differential equations in terms of the generalized forces and displacements which are convenient for the numerical solution.

A review of results in the theory of laminated rods, plates and shells which takes account of interlayer defects and imperfections in the contact between the layers of the delamination type has been given in [1]. The majority of papers published on this problem have been concerned with investigating stability and edge effects. The stress–strain state has not been studied sufficiently. In this paper, a model is proposed which enables us to extend the theory of three-layer, thin-walled structures with ideal adhesion between the layers [2] to the case when the contact between the layers is non-ideal.

1. INITIAL ASSUMPTIONS

The static loading of non-uniformly heated, three-layer rods with thin elastic isotropic layers of different thickness and a transversely isotropic filler is considered. Bernoulli's hypotheses are assumed to hold in the case of the outer layers. In accordance with the terminology established in the theory of three-layer structures, the filler is assumed to be rigid (carrying longitudinal forces and moments) and, also, transmits transverse shear and transverse normal strains and stresses. It is assumed that there are defective regions which are simulated by non-propagating shear layer separations for which, in the surfaces of contact between the outer layers and the filler, a discontinuity solely in the longitudinal components of the displacement vector is characteristic [3].

The rod is assumed to be subjected to transverse loads $q_1(x)$ and $q_2(x)$ (1 and 2 are the numbers of the external supporting layers) which are distributed over the surfaces of the carrying layers and the rod is non-uniformly heated to a specified temperature $T(x, z)$ throughout its length and thickness.

Henceforth, x is the longitudinal coordinate of the rod ($0 < x < l$), z is the transverse coordinate, h_1, h_2, $h_3 = 2c$ are the thicknesses of the first and second supporting layers and of the filler respectively, and $\Delta x^k = x_2^k - x_1^k$ ($k = 1, 2$) is the length of the layer separation (Fig. 1).

Furthermore, the following assumptions are made which define the model employed for the deformation of a three-layer rod with separation of the layers.

1. The normal displacements throughout the thickness of the filler are distributed linearly [4]

$$w^3 = w + zc^{-1}v \tag{1.1}$$

where $w = \frac{1}{2}(w^1 + w^2)$ is the mean deflection of the filler, w^k is the deflection of the kth layer ($k = 1, 2, 3$) and $v = \frac{1}{2}(w^1 - w^2)$ is a function which characterizes the compression of the filler. In particular, $w^1 = w^2$ and $v = 0$ for an incompressible filler.

2. Continuity of the longitudinal and normal displacements on passing from one layer to another is ensured in domains where there is ideal contact between the layers.

†*Prikl. Mat. Mekh.* Vol. 59, No. 3, pp. 475–484, 1995.

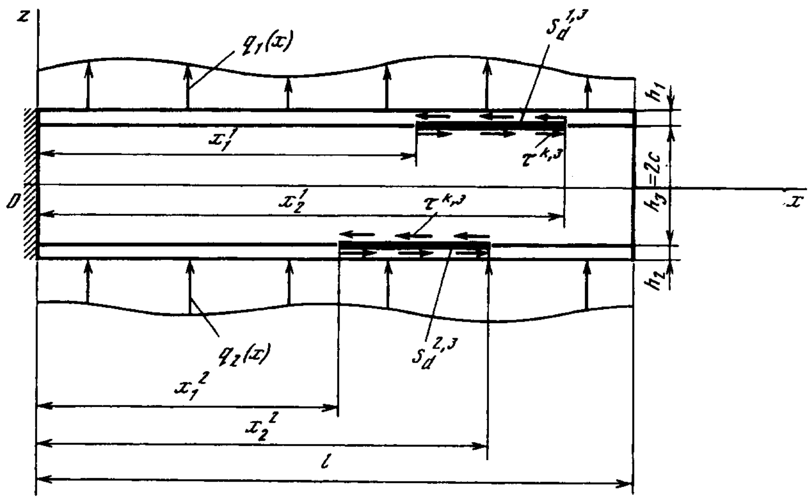

Fig. 1.

3. The normal and longitudinal displacements are assumed to be continuous throughout the thickness of the three-layer stack over the whole of the domain $0 < x < l$ only at the initial stage of deformation until the shear stresses $\hat{\sigma}_{xz}^{k3}$ ($k = 1, 2$), which, in the domains where $[\sigma_{xz}^{k3}]$ delamination occurs, are transmitted from the carrying layers to the filler, exceed the limiting permissible values of for the filler material which are determined using the Coulomb–Amonton law

$$\sigma_{xz}^{k,3} \leqslant [\sigma_{xz}^{k,3}] = \hat{\sigma}_{xz}^{k,3} + f_k \sigma_z^3 \tag{1.2}$$

In (1.2), $\hat{\sigma}_{xz}^{k3}$ are the limiting values of the shear stresses for the given material that characterize the shear resistance of the contacting layers, which is independent of the normal pressure, σ_z^3 are the normal stresses on the boundary of contact between the filler and the supporting layers, and f_k are the friction coefficients for the edges of the layer separations. Some estimates of $\hat{\sigma}_{xz}^{k3}$ for various materials are given in [5].

When the interlayer shear stresses exceed the permissible value, a slip occurs between the layers (the relative displacements of the layers in domains of layer separation) and interlayer shear forces $\tau^{k,3}$, $k = 1, 2$ manifest themselves (Fig. 1).

4. By analogy with the theory of composite rods with elastic-compliant shear links [6], it is assumed, in view of the smallness of the deformations, that the interlayer shear forces are proportional to the relative displacement of the layers in regions of layer separation. Then, in the case of the positive direction of loading which has been adopted, the quantities $\tau^{k,3}$ will be positive if they are directed as shown in Fig. 1 and they are equal to

$$\tau^{k,3}(x) = \lambda_k (u^k - u^3)\big|_{z=(-1)^{k+1}c}, \quad x_1^k \leqslant x \leqslant x_2^k \tag{1.3}$$

where λ_k are specified constants which are similar in their meaning to the coefficients of rigidity of interfaces in the theory of composite rods.

Relations (1.3) can be extended to the case when there is a non-linear relation between the forces $\tau^{k,3}$ and the relative displacements of the layers in regions of layer separation.

2. DISPLACEMENTS, STRAINS AND STRESSES

On integrating Cauchy's relation for the transverse shear strains in the filler $\varepsilon_{xz}^3 = \partial u^3/\partial z + \partial w^3/\partial x$ and taking account of the linear distribution of the normal displacements throughout the thickness of the central layer (1.1) which has been assumed and the conditions of continuity of the displacements on the boundaries of contact of the layers, and, also, on introducing u_1 and u_2 as additional unknown displacements of the carrying layers in the regions of layer separation, the distribution of the longitudinal and normal displacements in the layers of a three-layer rod can be represented in the following invariant form the normal displacements when $x \in [0, l]$

$$
\begin{aligned}
&w^1 = w + \upsilon \qquad && (-c \leqslant z \leqslant c + h_1)\\
&w^2 = w - \upsilon && (-c - h_2 \leqslant z \leqslant -c)\\
&w^3 = w + zc^{-1}\upsilon && (-c \leqslant z \leqslant c)
\end{aligned}
\tag{2.1}
$$

the longitudinal displacements when $x \in [0, l]$ (the prime denotes the derivatives with respect to x)

$$
\begin{aligned}
&u^k = u_k - [z + (-1)^k c][w - (-1)^k \upsilon]' \quad (k = 1, 2)\\
&u^3 = u + z(\alpha^3 - w' - \tfrac{1}{2} z^2 c^{-1} \upsilon')
\end{aligned}
\tag{2.2}
$$

Here u are the longitudinal displacements of the neutral layer of the filler and $\alpha^3 = \varepsilon_{xz}^3$ is the angle of transverse shear in the filler. In regions where there is ideal contact between the layers S_c, the displacements u^k are given by the formulae [4]

$$
u_k = u^3 |_{z=(-1)^{k+1}c} = u - (-1)^k c(\alpha^3 - w') - \tfrac{1}{2} c\upsilon' \quad (k = 1, 2)
\tag{2.3}
$$

The longitudinal components of the total strain of the layers and the transverse shear strains and compression in a transverse direction for the filler in the whole of the domain have the form

$$
\begin{aligned}
&\varepsilon_x^k = u_k' - [z + (-1)^k c][w - (-1)^k \upsilon]'' \quad (k = 1, 2)\\
&\varepsilon_x^3 = u' + z(\alpha^3 - w') - \tfrac{1}{2} z^2 c^{-1} \upsilon''\\
&\varepsilon_{xz}^3 = \alpha^3, \quad \varepsilon_z^3 = \upsilon c^{-1}
\end{aligned}
\tag{2.4}
$$

By the Hooke–Neumann law, the normal stresses in the carrying isotropic layers are determined by the expressions

$$
\sigma_x^k = E_k(\varepsilon_x^k - \alpha_k T) \quad (k = 1, 2)
\tag{2.5}
$$

where E_k and α_k are the modulus of elasticity and the coefficient of linear thermal expansion of the kth carrying layer and $T(x, z)$ is the specified temperature.

The thermoelasticity relations for a transversely isotropic compressible filler can be represented in the form

$$
\begin{aligned}
&\sigma_x^3 = E_3(\nu_{11}\varepsilon_x^3 + \nu_{12}\varepsilon_z^3 - \nu_{1T}\alpha_3 T)\\
&\sigma_z^3 = E_z^3(\nu_{11}\varepsilon_z^3 + \nu_{13}\varepsilon_x^3 - \nu_{2T}\alpha_3 T)\\
&\sigma_{xz}^3 = G_3\alpha^3, \quad \nu_{11} = 1/(1 - \nu_1\nu_2)\\
&\nu_{12} = \nu_1/(1 - \nu_1\nu_2), \quad \nu_{13} = \nu_2/(1 - \nu_1\nu_2)\\
&\nu_{1T} = \nu_{11} + \nu_{12}\alpha_z^3/\alpha_3, \quad \nu_{2T} = \nu_{12} + \nu_{11}\alpha_z^3/\alpha_3
\end{aligned}
\tag{2.6}
$$

Here, E_3 and α_3 are the modulus of elasticity and the coefficient of linear thermal expansion in the isotropic plane, G_3 is the shear modulus, E_z^3 is the modulus of elasticity of the filler in the transverse direction and ν_1 and ν_2 are Poisson's ratios of the filler material which characterize the reduction in the dimensions in the plane of isotropy when the material is stretched in a transverse direction and the reduction in the dimensions in the transverse direction when the material is stretched in the plane of isotropy, and α_z^3 is the coefficient of linear thermal expansion of the filler material in the transverse direction.

3. THE EQUATIONS OF EQUILIBRIUM AND BOUNDARY CONDITIONS

The equations of equilibrium of a three-layer bar with separations of the layers under bending can be obtained from the extended Lagrange variational equation

$$\delta Э = \delta\Pi - \delta A_q - \delta A_\tau = 0 \tag{3.1}$$

in which, apart from the variation in the potential energy of deformation of the rod $\delta\Pi$ and the variation of the work of the external surface load δA_q, the variation in the work of the interlayer shear forces $\tau^{k,3}$ for the corresponding relative displacements of the layers in the delamination regions, δA_τ, is also taken into account.

The potential energy of deformation of a three-layer bar, taking into account its transverse shears and compression in the filler, is equal to

$$\Pi = b\int_0^l\left[\int_c^{c+h_1}\sigma_x^1\varepsilon_x^1 dz + \int_{-c-h_2}^{-c}\sigma_x^2\varepsilon_x^2 dz + \int_{-c}^{c}\left(\sigma_x^3\varepsilon_x^3 + \sigma_{xz}^3\varepsilon_{xz}^3 + \sigma_z^3\varepsilon_z^3\right)dz\right]dx \tag{3.2}$$

Using relationships (2.4)–(2.6), we can reduce the expression for the potential energy of deformation to the form

$$\Pi = \int_0^l\left[\hat{N}u' + \hat{H}(\alpha^3)' - \hat{M}w'' - \hat{L}\upsilon'' + Q_x^3\alpha^3 + c^{-1}Q_z^3\upsilon\right]dx + \sum_{k=1}^{2}\int_{S_d^{k,3}} N^k u_k' dx \tag{3.3}$$

Here

$$\hat{N} = N - N^d, \quad \hat{H} = H - H^d, \quad \hat{M} = M - M^d, \quad \hat{L} = L - L^d \tag{3.4}$$

are the total specific forces and moments in a three-layer bar with separations of the layers.

The quantities **N**, **H**, **M** and **L**, occurring on the right-hand side of formulae (3.4), are the corresponding force factors in a three-layer rod when there is no separation of the layers [4]

$$N = \sum_{k=1}^{3} N^k, \quad H = M^3 + c(N^1 - N^2)$$
$$M = \sum_{k=1}^{3} M^k + c(N^1 - N^2), \quad L = M^1 - M^2 + \tfrac{1}{2}c(N^1 + N^2) + G^3 \tag{3.5}$$

where N is the total longitudinal force in a three-layer rod, M is the total bending moment in the three-layer rod with respect to the neutral line of the filler, H is the bending moment, which is determined by the transverse shear deformations in the filler (the shear moment) and L is the bending moment caused by taking the compressibility of the filler into account.

In this case, the specific forces and moments in the separate layers of a three-layer bar are

$$N^1 = b\int_c^{c+h_1}\sigma_x^1 dz, \quad N^2 = b\int_{-c-h_2}^{-c}\sigma_x^2 dz, \quad N^3 = b\int_{-c}^{c}\sigma_x^3 dz$$
$$M^1 = b\int_c^{c+h_1}\sigma_x^1(z-c)dz, \quad M^2 = b\int_{-c-h_2}^{-c}\sigma_x^2(z+c)dz, \quad M^3 = b\int_{-c}^{c}\sigma_x^3 z dz \tag{3.6}$$
$$G^3 = \frac{b}{2c}\int_{-c}^{c}\sigma_x^3 z^2 dz, \quad Q_x^3 = b\int_{-c}^{c}\sigma_{xz}^3 dz, \quad Q_z^3 = b\int_{-c}^{c}\sigma_z^3 dz$$

Here, G^3 is the additional second-order bending moment which appears when the compressibility of the filler is taken into account, Q_x^3 and Q_z^3 are transverse forces which are sensed by the filler and caused by the transverse shear and the transverse normal strains in the central layer, respectively.

The additional force factors N^d, H^d, M^d, L^d in (3.4), which occur in a three-layer rod due to defects in the surfaces of contact between layers, have a different form depending on the number and the relative arrangement of the delamination.

The final expressions for the total forces and moments for the various ways of arranging the layer separations with respect to one another are determined by the formulae:
in the region of a first isolated layer separation $S_d^{1,3}$

$$\hat{N} = N^2 + N^3, \quad \hat{H} = M^3 - cN^2$$

$$\hat{M} = \sum_{k=1}^{3} M^k - cN^2, \quad \hat{L} = M^1 - M^2 + G^3 + \tfrac{1}{2}cN^2 \tag{3.7}$$

in the region of a second isolated separation of the layers $S_d^{2,3}$

$$\hat{N} = N^1 + N^3, \quad \hat{H} = M^3 + cN^1$$

$$\hat{M} = \sum_{k=1}^{3} M^k + cN^1, \quad \hat{L} = M^1 - M^2 + G^3 + \tfrac{1}{2}cN^1 \tag{3.8}$$

and in the region of overlap of the layer separations $S_d^{1,3} \forall S_d^{2,3}$

$$\hat{N} = N^3, \quad \hat{H} = M^3, \quad \hat{M} = \sum_{k=1}^{3} M^k, \quad \hat{L} = M^1 - M^2 + G^3 \tag{3.9}$$

In the limiting case of a three-layer rod with ideal adhesion between the layers (taking account of transverse shears and the compressibility of the filler), the total forces and moments will correspond to those introduced in [4].

On integrating expression (3.3) by parts and analysing the terms outside the integral, we find that the generalized forces and displacement of a three-layer rod with delamination must be chosen as follows:

N^1	N^2	$\hat{N}$	$\hat{H}$	$\hat{M}$	$\hat{L}$	$\hat{M}'$	$\hat{L}'$
u_1	u_2	u	α^3	w'	υ'	w	υ

Note that the unknown interlayer shear forces $\tau^{k,3}$ due to interaction between the layers in regions where there is separation of the layers are determined in terms of the longitudinal displacements on the boundaries of contact of the defective regions using formulae (1.3). Hence, the number of basic unknowns in the problem of the bending of a three-layer rod with layer separation defects is equal to 16.

The work A_q of the external distributed surface loads $q_1(x)$ and $q_2(x)$ which are applied to the surface $z = c + h_1$ and $z = -c + -h_2$, respectively (see Fig. 1), and the work A_τ of the interlayer contact forces in regions where layer separation occurs are equal to

$$A = A_q + A_\tau = \int_0^l (q_1 w^1 + q_2 w^2)dx + \sum_{k=1}^{2} \int_{S_d^{k,3}} \tau^{k,3}(u^k - u^3)\Big|_{z=(-1)^{k+1}c} dx =$$

$$= \int_0^l [(q_1 + q_2)w + (q_1 - q_2)\upsilon]dx +$$

$$+ \sum_{k=1}^{2} \int_{S_d^{k,3}} \left\{ u_k - u + (-1)^k \frac{c}{2}\left[\alpha^3 - (-1)^k w' + \frac{\upsilon'}{2}\right]\right\} dx \tag{3.10}$$

(relationships (1.3), (2.1)–(2.3) and (2.6) have been taken into account here). On varying Π, A_q and A_τ while taking account of the expressions for the total and generalized moments and forces in the different regions of the bar according to (3.7)–(3.9), we obtain an extended Lagrange variational equation in the form

$$\delta Э = -\int_{S_c} [N'\delta u + (H' - Q_x^3)\delta\alpha^3 + (M'' + q_1 - q_2)\delta w +$$

$$+(L'' - c^{-1}Q_z^3 + q_1 - q_2)\delta\upsilon]dx - \sum_{k=1}^{2} \int_{S_d^{k,3}} \{((N^k)' + \tau^{k,3})\delta u_k + ((N^3)' + (N^{3-k})' -$$

$$-\tau^{k,3})\delta u+[\hat{H}'+(-1)^k\tau^{k,3}-Q_x^3]\delta\alpha^3+[\hat{H}''+(-1)^k c(\tau^{k,3})'+q_1+q_2]\delta w+$$

$$+[\hat{L}''-\tfrac{1}{2}c(\tau^{k,3})'-c^{-1}Q_z^3+q_1-q_2]\delta\upsilon\}dx+$$

$$+(N\delta u+H\delta\alpha^3-M\delta w'+M'\delta w-L\delta\upsilon'+L'\delta\upsilon)|_{S_c}+$$

$$+\sum_{k=1}^{2}\{N^k\delta u_k+\hat{N}\delta u+\hat{H}\delta\alpha^3-\hat{M}\delta w'+[\hat{M}'-(-1)^k c\tau^{k,3}]\delta w-$$

$$-\hat{L}\delta\upsilon'+(\hat{L}'+\tfrac{1}{2}c\tau^{k,3})\delta\upsilon\}|_{x_1^k}^{x_2^k}=0 \qquad (3.11)$$

where $S_d^{k,3}$ are the domains where there is layer separation ($k = 1, 2$) and S_c are the defect-free domains.

By virtue of the arbitrariness in the variations of the displacements, the equilibrium equations for the various regions of the rod (defect-free regions and regions with layer separation) and, also, the conditions at the ends of the bar and the matching conditions on the boundaries of contact between the defect-free and defective regions follow from the extended Lagrange variational equation.

In the general case when the regions of layer separation are arbitrarily located with respect to one another, the systems of equilibrium equations for the defect-free regions, for regions with isolated layer separations and for the region with overlapping layer separations must be simultaneously solved while taking account of the corresponding end conditions and the matching conditions.

By introducing the unit step functions

$$\Delta H^k=H^k(x-x_1^k)-H^k(x-x_2^k)=\begin{cases}1, & x\in[x_1^k;\ x_2^k]\\ 0, & x\in[0;\ x_1^k[\cup]x_2^k;l]\end{cases}$$

and when account is taken of relations (3.4), the systems of equations for the equilibrium of the forces and moments in the different domains can be represented in the invariant form

$$\begin{aligned}
&\hat{N}'-\sum_{k=1}^{2}\tau^{k,3}\Delta H^k=0,\quad ((N^k)'+\tau^{k,3})\Delta H^k=0\quad (k=1,2)\\
&\hat{H}'-Q_x^3-c\sum_{k=1}^{2}(-1)^{k+1}\tau^{k,3}\Delta H^k=0\\
&\hat{M}''-c\sum_{k=1}^{2}(-1)^{k+1}(\tau^{k,3})'\Delta H^k+q_1+q_2=0\\
&\hat{L}''-c_z^{-1}Q_z^3-\tfrac{1}{2}c\sum_{k=1}^{2}(\tau^{k,3})'\Delta H^k+q_1-q_2=0
\end{aligned}\qquad (3.12)$$

The forces and moments in the different domains occurring in the equations are governed by (3.7)–(3.9), taking (2.4) and (2.5), (3.5) and (3.6) into account.

Each of the terms outside the integral in the variational equation (3.11) can be represented in the form

$$F_1(x)\delta U|_0^{x_1^k}+F_2(x)\delta U|_{x_1^k}^{x_2^k}+F_3(x)\delta U|_{x_2^k}^{l}=$$

$$=\{-F_1(0)+[F_1(x_1^k)-F_2(x_1^k)]+[F_2(x_2^k)-F_3(x_2^k)]+F_3(l)\}\delta U$$

where $F_i(x)$ is a generalized force factor and U is the generalized displacement corresponding to it. The end conditions for the rod, when $x = 0, l$ and the conditions for the matching of the solutions of the systems of equations when $x = x_1^k$ and $x = x_2^k$, which describe the stress–strain state of the rod in adjacent domains, follow therefore from the variational equation (3.11) and the condition that the variations δU are arbitrary.

The natural static boundary conditions when $x = 0, l$ are written in the general case as

$$\hat{N}=\hat{H}=\hat{M}=\hat{L}=\hat{M}'=\hat{L}'=0 \qquad (3.13)$$

If the ends of the rod are in defect-free regions, as shown in Fig. 1, then, under conditions (3.13), the carets over the corresponding quantities are superfluous.

If the force factors are non-zero, the corresponding kinematic end conditions will be

$$u = \upsilon = w = \alpha^3 = w' = \upsilon' = 0 \tag{3.14}$$

Moreover, according to (1.3), the following conditions must be satisfied at the vertices of the layer separations

$$\tau^{k,3} = 0 \quad \text{when} \quad x = x_1^k, x_2^k \quad (k = 1, 2) \tag{3.15}$$

The conditions for the matching of the solution when $x_1^1 = x_1^2, x_2^1 = x_2^2$ have the form

$$\begin{aligned} &N = N^3, \quad H = M^3, \quad M = \sum_{k=1}^{3} M^k \\ &M' = \sum_{k=1}^{3} (M^k)' + c(\tau^{1,3} - \tau^{2,3}), \quad L = M^1 - M^2 + G^3 \\ &L' = (M^1 - M^2 + G^3)' + \tfrac{1}{2}c(\tau^{1,3} + \tau^{2,3}) \end{aligned} \tag{3.16}$$

The forces and moments in the defect-free regions when $x \to x_1^k - 0$ and $x \to x_2^k + 0$ are shown on the left-hand sides of equalities (3.16) while the forces and moments in the domains of layer separation when $x \to x_1^k + 0$ and $x \to x_2^k - 0$, respectively, are shown on the right-hand sides of these equalities.

4. ELASTICITY RELATIONS

In view of the complexity of the overall system of elasticity relations we shall confine ourselves to the case when the regions of layer separation are completely identical. Then, taking relationships (2.4)–(2.6), (3.4)–(3.6) and (3.9) into account, we obtain:

for a defect-free region

$$\begin{aligned} &\hat{N} = Ehb[c_{16}u' + \tfrac{1}{2}h(c_{12}(\alpha^3)' - c_{15}w'' - c_{26}\upsilon'') + 2h^{-1}\gamma_3 t_3^{-1}\nu_{12}\upsilon] - N_t \\ &\hat{H} = \tfrac{1}{2}Eh^2b[c_{12}u' + \tfrac{1}{2}h(c_{36}(\alpha^3)' - c_{46}w'' - c_{24}\upsilon'')] - H_t \\ &\hat{M} = \tfrac{1}{2}Eh^2b[c_{15}u' + \tfrac{1}{2}h(c_{46}(\alpha^3)' - c_{66}w'' - c_{45}\upsilon'')] - M_t \\ &\hat{L} = \tfrac{1}{2}Eh^2b[c_{26}u' + \tfrac{1}{2}h(c_{24}(\alpha^3)' - c_{45}w'' - c_{61}\upsilon'') + \tfrac{1}{3}h^{-1}\gamma_3\nu_{12}\upsilon] - L_t \\ &E = h_k^{-1}\sum_{k=1}^{3} E_k h_k, \quad \gamma_k = E_k h_k / Eh, \quad t_k = h_k / h \\ &c_{12} = t_3(\gamma_1 - \gamma_2), \quad c_{13} = \gamma_1 t_1 - \gamma_2 t_2 \\ &c_{15} = c_{12} + c_{13}, \quad c_{16} = \gamma_1 + \gamma_2 + \nu_{11}\gamma_3 \\ &c_{23} = t_3(\gamma_1 t_1 + \gamma_2 t_2), \quad c_{24} = t_3(c_{13} + \tfrac{1}{2}c_{12}) \\ &c_{26} = t_3^{-1}(c_{23} + \tfrac{1}{2}c_{36}), \quad c_{33} = \tfrac{4}{3}(\gamma_1 t_1^2 + \gamma_2 t_2^2) \\ &c_{34} = \tfrac{4}{3}(\gamma_1 t_1^2 - \gamma_2 t_2^2) + \tfrac{1}{3}t_3 c_{13}, \quad c_{36} = t_3^2(\gamma_1 + \gamma_2 + \tfrac{1}{3}\nu_{11}\gamma_3) \\ &c_{45} = c_{24} + c_{34}, \quad c_{46} = c_{36} + c_{23} \\ &c_{61} = c_{23} + c_{33} + \tfrac{1}{4}t_3^2(\gamma_1 + \gamma_2 + \tfrac{1}{5}\gamma_3\nu_{11}), \quad c_{66} = c_{36} + 2c_{23} + c_{33} \end{aligned} \tag{4.1}$$

where E is the reduced modulus of elasticity of the three-layer stack, γ_k is the relative dimensionless stiffness to stretching of the kth layer and t_k is the relative thickness of the kth layer [2] ($k = 1, 2, 3$);

in regions where layer separation occurs

$$\begin{aligned}
&\hat{N} = Ehb(\gamma_3\nu_{11}u' - \tfrac{1}{2}hc_{64}\nu_{11}\upsilon'' + 2h^{-1}t_3^{-1}\gamma_3\nu_{12}\upsilon) - T_3 \\
&\hat{H} = Eh^3bc_{63}((\alpha^3)' - w'') - m_3 \\
&\hat{M} = \tfrac{1}{2}Eh^2b[\gamma_1t_1u_1' - \gamma_2t_2u_2' + \tfrac{1}{2}h(c_{63}(\alpha^3)' - c_{56}w'' - c_{62}\upsilon'')] - \sum_{k=1}^{3} m_k \\
&\hat{L} = \tfrac{1}{2}Eh^2b[\gamma_1t_1u_1' + \gamma_2t_2u_2' + c_{64}u' - \tfrac{1}{2}h(c_{62}w'' + c_{56}\upsilon'' + \tfrac{1}{3}h^{-1}\gamma_3\nu_{12}\upsilon] - (m_1 - m_2 + m_4) \\
&N^1 = Ehb\gamma_1[u_1' - \tfrac{1}{2}ht_1(w+\upsilon)''] - T_1 \\
&N^2 = Ehb\gamma_2[u_2' - \tfrac{1}{2}ht_2(w-\upsilon)''] - T_2 \\
&c_{56} = c_{33} + \tfrac{3}{5}c_{63}, \quad c_{62} = \tfrac{4}{3}(\gamma_1t_1^2 - \gamma_2t_2^2) \\
&c_{63} = \tfrac{1}{12}\gamma_3t_3^2\nu_{11}, \quad c_{64} = \tfrac{1}{6}\gamma_3t_3\nu_{11}
\end{aligned} \tag{4.2}$$

Here, the transverse forces in the filler for the entire region of the rod are

$$\begin{aligned}
&Q_x^3 = G_3hbt_3\alpha^3 \\
&Q_z^3 = E_z^3hb\left(2h^{-1}\nu_{11}\upsilon + t_3\nu_{13}u' - \tfrac{1}{12}t_3^2\nu_{13}\upsilon'' - h^{-1}\alpha_3\nu_{2T}\int_{-c}^{c} Tdz\right)
\end{aligned} \tag{4.3}$$

In relations (4.1)–(4.3), the last terms, which characterize the effect of temperature on the stress–strain state of the three-layer rod under bending have the form

$$\begin{aligned}
&N_t = \sum_{k=1}^{3} T_k, \quad H_t = c(T_1 - T_2) + m_3 \\
&M_t = \sum_{k=1}^{3} m_k + c(T_1 - T_2), \quad L_t = m_1 - m_2 + m_4 + \tfrac{1}{2}c(T_1 + T_2)
\end{aligned} \tag{4.4}$$

where

$$T_k = Eb\frac{\alpha_k\gamma_k}{t_k}\int_{h_k} Tdz \quad (k = 1,2), \quad T_3 = Eb\nu_{1T}\frac{\alpha_3\gamma_3}{t_3}\int_{-c}^{c} Tdz$$

$$m_1 = Eb\frac{\alpha_1\gamma_1}{t_1}\int_{c}^{c+h_1} T(z-c)dz, \quad m_2 = Eb\frac{\alpha_2\gamma_2}{t_2}\int_{-c-h_2}^{-c} T(z+c)dz$$

$$m_3 = Eb\nu_{1T}\frac{\alpha_3\gamma_3}{t_3}\int_{-c}^{c} Tzdz, \quad m_4 = Eb\nu_{1T}\frac{\alpha_3\gamma_3}{t_3}\int_{-c}^{c} Tz^2dz$$

5. RESOLVENT SYSTEMS OF EQUATIONS

When account is taken of the elasticity relations (4.1)–(4.4), the systems of equilibrium equations (3.12) can be written in terms of the notation adopted for the generalized forces and displacements in the Cauchy normal form which is convenient for numerical solution.

The elasticity relations for the case when the domains of layer separation overlap, can be represented, after some reduction, in vector matrix form

$$A\mathbf{X} = \mathbf{d}$$

$$\mathbf{X}^T = \left\| u_1', u_2', u', (\alpha^3)', w'', \upsilon'' \right\|^T \tag{5.1}$$

$$\mathbf{d} = \left\| \begin{matrix} -(N^1 + T_1) \\ N^2 + T_2 \\ -(\hat{N} + T_3 - \eta v) \\ -(\hat{H} + m_3) \\ \hat{M} + \sum_{k=1}^{3} m_k \\ \hat{L} + m_1 - m_2 + m_4 - \frac{1}{2} t_3 \eta v \end{matrix} \right\|, \quad \eta = 2Ebt_3^{-1} v_{12} \gamma_3$$

where A is the symmetric matrix of the stiffness coefficients and $\mathbf{X}^T$ is the transposed vector of the unknowns.

On solving system (5.1), we find the vector of the unknowns in the form

$$\mathbf{X} = B\mathbf{d} \tag{5.2}$$

The components of the matrices A and B and the formulae for the isolated and defect-free regions are not presented here on account of their length.

On adopting the notation

$$y_1 = N^1, \quad y_2 = u_1, \quad y_3 = N^2, \quad y_4 = u_2$$
$$y_5 = \hat{N}, \quad y_6 = u, \quad y_7 = \hat{H}, \quad y_8 = \alpha^3$$
$$y_9 = \hat{M}, \quad y_{10} = w', \quad y_{11} = \hat{M}', \quad y_{12} = w$$
$$y_{13} = \hat{L}, \quad y_{14} = v', \quad y_{15} = \hat{L}'', \quad y_{16} = v$$

for the generalized forces and displacements, we obtain the resolvent systems of equations for the bending of a three-layer rod in the normal Cauchy form for the different domains

$$y_i' = f_i(y_j, q_1, q_2, T) \tag{5.3}$$

where $i, j = 1, 2, \ldots, 16$ for the region of the rod with layer separation and $i, j = 1, 2, \ldots, 12$ for the defect-free regions.

The solution of the systems of equations (5.3) can be obtained using any stable numerical method such as the method of orthogonal pivotal condensation, for example.

Note that the model of the deformation of non-uniformly heated three-layer rods which has been presented can be extended in a natural way to the case of plates and shallow shells. In this case, the number of basic unknowns increases in the region of layer separation from 16 in the one-dimensional case to 24 in the two-dimensional case.

REFERENCES

1. GRIGOLYUK E. I., KOGAN Ye. A. and MAMAI V. I., Problems of the deformation of thin-walled laminated structures with layer separation. *Ross. Akad. Nauk, MTT* No. 2, 6–32, 1994.
2. GRIGOLYUK E. I. and CHULKOV P. P., *Stability and Vibrations of Three-layer Shells.* Mashinostroyeniye, Moscow, 1973.
3. PELEKH B. L., MAKSIMUK A. V. and KOROVAICHUK I. M., *Contact Problems for Laminated Structural Elements and Bodies with Coverings*. Naukova Dumka, Kiev, 1988.
4. IVANOV A. V. and CHULKOV P. P., Allowance for transverse deformations of the filler in problems of the stability of three-layer plates with different carrying layers. *Izv. Akad. Nauk SSSR, MTT* 6, 101–107, 1969.
5. KRAGEL'SKII I. V., DOBYCHIN M. N. and KOMBALOV V. S., *Fundamentals of the Analysis of Friction and Wear.* Mashinostroyeniye, Moscow, 1977.
6. RZHANITSYN A. R., *Theory of Composite Rods in Structural Design*. Stroiizdat, Moscow, 1948.

Translated by E.L.S.